Exercise 38

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{\sin^2 x}{x^2+1}$$

Solution

Multiply the numerator and denominator by the reciprocal of the highest power of x in the denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sin^2 x}{x^2+1} &= \lim_{x\to\infty} \frac{\sin^2 x}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{\frac{\sin^2 x}{x^2}}{(x^2+1)\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{\frac{\sin^2 x}{x^2}}{1+\frac{1}{x^2}} \\
&= \frac{\displaystyle\lim_{x\to\infty} \frac{\sin^2 x}{x^2}}{\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x^2}\right)} \\
&= \frac{\displaystyle\lim_{x\to\infty} \left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)}{\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x^2}\right)} \\
&= \frac{\left(\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}\right)\left(\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}\right)}{\displaystyle\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x^2}} \\
&= \frac{(0)(0)}{1+0} \\
&= 0
\end{aligned}$$